Advanced Systems Theory

26/01/2018, Friday, 14:00 – 17:00

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1 $(6+4+8=18$ pts) **Observability and detectability**

Consider the system Σ:

$$\dot{x}(t) = Ax(t),\ y(t) = Cx(t)$$

with state space $\mathbb{R}^n$ and output space $\mathbb{R}^p$.

(a) Assume that the system is observable. Show that if $y(t) = 0$ for all $t \geqslant 0$ then $x(t) = 0$ for all $t \geqslant 0$.

(b) Let $G \in \mathbb{R}^{n\times p}$. Show that x and y satisfy the equations of the system Σ if and only if they satisfy the equations

$$\dot{x}(t) = (A - GC)x(t) + Gy(t),\ y(t) = Cx(t).$$

(c) Now assume that Σ is detectable. Use the result in (b) to show that if $y(t) = 0$ for all $t \geqslant 0$ then $x(t) \to 0$ as $t \to \infty$.

2 $(4+5+5+4=18$ pts) **Stabilizability subspaces**

Consider the system $\dot{x}(t) = Ax(t) + Bu(t)$ with state space $\mathcal{X} = \mathbb{R}^n$. A subspace $\mathcal{V}$ of $\mathcal{X}$ is called a *stabilizability subspace* if for every $x_0 \in \mathcal{V}$ there exists an input function u such that $x_u(t, x_0) \in \mathcal{V}$ of all $t \geqslant 0$, and $\lim_{t\to\infty} x_u(t, x_0) = 0$.

(a) Let $\mathcal{V}$ be a stabilizability subspace. Show that $\mathcal{V}$ is controlled invariant.

(b) Show that the sum of two stabilizability subspaces is again a stabilizability subspace.

(c) Let $\mathcal{K}$ be a given subspace of $\mathcal{X}$. Show that there exists a largest stabilizability subspace $\mathcal{V}_s^*(\mathcal{K})$ contained in $\mathcal{K}$.

(d) Let $\mathcal{V}^*(\mathcal{K})$ be the largest controlled invariant subspace contained in $\mathcal{K}$. Show that $\mathcal{V}_s^*(\mathcal{K}) \subset \mathcal{V}^*(\mathcal{K})$.

3 $(3+6+9=18$ pts) **Controlled invariant subspaces**

Consider the system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t).\end{aligned}$$

with state space $\mathbb{R}^n$, input space $\mathbb{R}^m$, and output space $\mathbb{R}^p$. For a controlled invariant subspace $\mathcal{V}$, the set of 'friends' are defined by $\mathcal{F}(\mathcal{V}) := \{F \in \mathbb{R}^{n\times m} \mid (A + BF)\mathcal{V} \subseteq \mathcal{V}\}$. Let $\mathcal{V}^*$ be the largest controlled invariant subspace that is contained in $\ker C$. Show that

(a) $(A + BF + GC)\mathcal{V}^* \subseteq \mathcal{V}^*$ for every $F \in \mathcal{F}(\mathcal{V}^*)$ and $G \in \mathbb{R}^{n\times p}$.

(b) the unobservable subspace $\langle \ker C \mid A\rangle$ is contained in $\mathcal{V}^*$.

(c) there exists $F \in \mathcal{F}(\mathcal{V}^*)$ such that $\langle \ker C \mid A\rangle \subseteq \ker F$.

4 $(6 + 12 = 18$ pts) **Disturbance decoupling with measurement feedback**

Consider the system

$$\dot{x} = Ax + Bu + Ed$$
$$y = Cx$$
$$z = Hx$$

with

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

(a) Show that the problem of disturbance decoupling (from d to z) by measurement feedback is solvable for this system.

(b) Compute a dynamic controller Γ that makes the system from d to z decoupled.

5 (18 pts) **Tracking and regulation**

Consider the system given as the interconnection of the exosystem

$$\dot{x}_1(t) = 0$$

and the control system

$$\dot{x}_2(t) = -x_2(t) + x_4(t) + x_1(t)$$
$$\dot{x}_3(t) = x_4(t)$$
$$\dot{x}_4(t) = x_2(t) + 3x_3(t) + 2x_4(t) + u(t)$$
$$y(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$
$$z(t) = -x_1(t) + x_2(t).$$

Construct a regulator.

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10 pts free